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Breaking of Permutation Symmetry and Diagonal Group Action: Nielsen Model and the Standard Model as Low-Energy Limit

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We point out that, on the basis of a theorem by L. Michel, every model based on the symmetry $G \otimes \cdots \otimes G$ has a generic low-energy limit the breaking to diagonal action, i.e., to G. This applies in particular to the proposal by Nielsen to consider such a model with G the standard model group.

1. INTRODUCTION

In a series of papers Nielsen and collaborators[2] have considered the possibility that the grand unification group is of the form

$$\begin{aligned}\mathscr{G}_0 &= G_1 \otimes G_1 \otimes \cdots \otimes G_2 \otimes \cdots \otimes G_s \\ &= (G_1)^{\otimes k_1} \otimes (G_2)^{\otimes k_2} \otimes \cdots \otimes (G_s)^{\otimes k_s}\end{aligned} \tag{1}$$

and that the low-energy limit corresponds to breaking of this symmetry to the diagonal action over the G_i spaces, i.e., to

$$\mathscr{G}_1 = G_1 \otimes G_2 \otimes \cdots \otimes G_s \tag{2}$$

Naturally, the G_i factors should then be $SU(n)$ groups.

We will not enter into the discussion of the Nielsen model, for which we refer the reader to his papers (see footnote 2); our point here is just to point out that, once a symmetry of the kind considered by Nielsen is assumed, solutions breaking it to diagonal action do necessarily exist, as a consequence of a theorem of Michel (1971)—actually motivated by $SU(3)$

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[2]See Nielsen (1991) for a recent review; for previous contributions see the references cited there; recent ones are Bennet *et al.* (1988), Nielsen and Brene (1989), and Nielsen (1989); see also Froggat and Nielsen (1991).

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models (Michel and Radicati, 1971, 1973) in particle physics—on the geometry of group actions.

2. SETTING THE (REDUCED) STAGE

Let us now state a precise setting for our considerations. It will be a simplification of Nielsen's, but clearly it contains its essential characteristics, and our remarks will obviously generalize to his full setting.

Let us consider a smooth potential $V(\xi)$ defined on $\Xi = M \times \cdots \times M = (M)^{\times n}$, where $M \subseteq R^m$. We will consider coordinates $x \in R^m$ on each copy of M; i.e., $\Xi = M_1 \times \cdots \times M_n$, $\xi = (x_1, \ldots, x_n)$, $x_i \in M_i \subseteq R^m$. The potential $V: \Xi \to R$ will be supposed symmetric under permutations of the x's, i.e.,

$$V(\sigma\xi) = V(\xi) \qquad \forall \sigma \in S \tag{3}$$

with S (a subgroup of) the permutation group on n elements S_n. In other words,

$$V(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = V(x_1, \ldots, x_n) \tag{4}$$

Moreover, we assume that an action of G is defined on M, where G is a compact Lie group, and V is also invariant under the action of $\mathcal{G} \equiv (G)^{\otimes n}$ on $(M)^{\times n}$, i.e., $\forall (g_1, \ldots, g_n) \in G \otimes \cdots \otimes G$ one has

$$V(g_1 x_1, \ldots, g_n x_n) = V(x_1, \ldots, x_n) \tag{5}$$

Finally, the full symmetry group of V is therefore

$$G_V = S \otimes_{\rightarrow} \mathcal{G} \tag{6}$$

where $\otimes_{\rightarrow}$ denotes semidirect product. We will refer to this situation as a Nielsen model.

To fix ideas, let us consider $M = C^2 \simeq R^4$, $G = SU(2)$, $n = 2$, so that $S = Z_2$; a possible $V(\xi)$ in this case would be

$$V(\xi) = (\lambda x_1^2 + x_1^4) + (\lambda x_2^2 + x_2^4) + \beta (x_1^2 - x_2^2)^2 \tag{7}$$

where λ and β are real parameters, $x_i \in R^4 \simeq C^2$, and $SU(2)$ acts in C^2 (in R^4) by its standard action (by the realification of its standard action).

3. MICHEL'S THEOREM: CRITICAL ORBITS

Let us now recall very briefly the content of Michel's theorem (Michel, 1971); for more details, generalizations, applications, etc., the reader is referred, e.g., to Michel (1972), Abud and Sartori (1983), Palais (1979), and Gaeta (1992*a*).

Given an action of G (a compact finite-dimensional Lie group) on the finite-dimensional smooth manifold M, to any point $x \in M$ we can associate the isotropy subgroup $G_x \subseteq G$:

$$G_x = \{ g \in G \mid gx = x \} \tag{8}$$

Points on the same G-orbit have conjugated isotropy subgroups:

$$y = gx \Rightarrow G_y = gG_x g^{-1} \tag{9}$$

If we consider conjugation classes of isotropy subgroups,

$$[G_x] = \{ H \subseteq G \mid H = gG_x g^{-1}, g \in G \}$$

clearly $y = gx \Rightarrow [G_y] = [G_x]$ and

$$[G_y] = [G_x] \Leftrightarrow y \simeq x \tag{10'}$$

defines an equivalence relation $\simeq$ not only in M, but also in the orbit space $\Omega = M/G$:

$$\omega \simeq \omega' \Leftrightarrow [G_x] = [G_y], \qquad x \in \omega, \qquad y \in \omega' \tag{10''}$$

The set of points $x \in M$ (respectively, of orbits $\omega \in \Omega$) in the same equivalence class under (10) is called a *stratum* in M (respectively, in Ω).

For a scalar potential $\Phi: M \to R$ invariant under G, $\Phi(gx) = \Phi(x)$, $\forall x \in M$, $\forall g \in G$, critical points will clearly appear in G-orbits, and we will talk of critical orbits. For a given G-action on M, there will be G-orbits which are critical for *any* smooth G-invariant potential; these will be called G-critical orbits. The simplest example of a G-critical orbit is given by the Z_2 action $x \to -x$ on the real line R: any even function $f(x) = f(-x)$ has zero gradient at the origin.

In this notation, Michel's theorem tells that: *A G-orbit is G-critical if and only if it is isolated in its stratum.*

This clearly requires that we define a topology in orbit space; this can be issued by a distance among orbits, which can be thought of as

$$d(\omega, \omega') = \min_{\{x \in \omega, y \in \omega'\}} \|x - y\| \tag{11}$$

where $\|x - y\|$ is the distance in M. We will not enter into details, since physically one wants $M = C^N \simeq R^{2N}$ and the distance is just the Euclidean one. (Moreover, our orbits isolated in their strata will actually belong to strata made of finite discrete sets of orbits.)

To fix ideas, let us consider some simple examples.

(i) $M = R$, $G = Z_2$, as noted before.

(ii) $M = S^2 \subset R^3$, $G = SO(2)$ acting as rotations around the vertical axis; the north and south poles are critical points for any G-invariant potentials, and indeed (distinct) G-critical orbits. The orbit space is in fact $\Omega = [-1, 1]$ (orbits are thought of as identified by the coordinate along the axis of rotation), with two strata, $\Omega_0 = \{-1\} \cup \{1\}$ and $\Omega_1 = (-1, 1)$.

(iii) $M = R^2$ with coordinates (x, y), $G = Z_2^x \otimes Z_2^y$ acting by $Z_2^x: (x, y) \to (-x, y)$; $Z_2^y: (x, y) \to (x, -y)$. The orbit space is $\Omega = R_+ \times R_+ = \{(x, y) | x \geq 0, y \geq 0\}$; there are four strata: $\Omega_0 = \{(0, 0)\}$, $\Omega_1^x = \{(0, y); y > 0\}$, $\Omega_1^y = \{(x, 0); x > 0\}$, $\Omega_2 = \{(x, y) | xy > 0\}$. Notice that although Ω_1^x and Ω_1^y correspond to isomorphic isotropy subgroups Z_2^x, Z_2^y, these are not conjugated in G, and therefore Ω_1^x, Ω_1^y are different strata.

4. MICHEL THEOREM AND LINEAR ACTIONS: CRITICAL DIRECTIONS

Notice that for linear actions of G on the linear space $M = R^m$ or $M = C^s$, strata in M will come on linear subspaces, so that there cannot be orbits (other than the trivial one made of the origin alone) $\omega \in \Omega = M/G$ isolated in their stratum. At most, one can have that there are orbits isolated in their stratum in the orbit space on $S^{m-1} \subseteq R^m$, i.e., $\Omega_s = S^{m-1}/G$; in other words, we can have at most one-dimensional strata. This is indeed enough for our purposes: in fact, it is easy to see that $\nabla V(x)$ is in the tangent space to the stratum of the G-orbit through x (Michel, 1971, 1972; Abud and Sartori, 1983); if this is one dimensional and $V(x) \to \infty$ for $|x| \to \infty$ (we also say such a potential is confining), we have that necessarily V has a critical orbit on this one-dimensional stratum (the modulus of this will depend on the parameters appearing in the potential). We express this fact by saying that (for a linear G-action) an orbit $\omega_s \in \Omega_s$ which is isolated in its stratum is a *critical direction*: any G-invariant confining potential has a critical orbit $\omega = \mu \omega_s$, $\mu \in R_+$.

If V is G invariant, confining, and depends on a real parameter λ (physically, $\lambda = 1/E$, so that $\lambda \geq 0$) such that for $\lambda = 0$, $\xi = 0$ is a minimum of $V(\xi)$ and that at some value $\lambda = \lambda_0 > 0$, $\xi = 0$ becomes unstable (i.e., a maximum or a saddle point for V), a straightforward consequence of the Michel theorem is the Equivariant Branching Lemma of Cicogna (1981) and Vanderbauwhede (1982).[3]

[3]Generalizations of this were given by Cicogna (1984; see also Cicogna and Degiovanni, 1984) for higher-dimensional critical space, by Golubitsky and Stewart (1985) for the Hopf bifurcation, and by Cicogna (1990) for nonlinear group actions. See also Golubitsky *et al.* (1988) and Gaeta (1990) for applications and Gaeta (1992*b*) for an extension to gauge symmetries.

The Equivariant Branching Lemma ensures that in this case $\forall\lambda$ there exists a critical orbit $\omega(\lambda)=\mu(\lambda)\omega_s$, and $\mu(\lambda)=0$ for $\lambda\leq\lambda_*$, where λ_* is the value at which $\xi=0$ loses stability in the ω_s direction (this can be different for different critical directions), while for $\lambda\geq\lambda_*$ there exists a smooth branch of critical points given by $(\lambda-\lambda_*)=s(\varepsilon)$, $\mu=\mu(\varepsilon)$.[4]

Such a result was actually already used in the pioneering work of Michel and Radicati (1971, 1973) on the octet of $SU(3)$ and in general on the adjoint action of $SU(n)$.

5. APPLICATION TO NIELSEN MODELS

Let us now see how this applies to Nielsen models, i.e., to the case that the group under consideration is $G_V=S\otimes_{\rightarrow}\mathscr{G}$, with $\mathscr{G}=(G)^{\otimes n}$, $S=S_n$ (see section 2).

The subgroups of G_V will be of the form $G_V^{\alpha,i}=S_\alpha\otimes_{\rightarrow}\mathscr{G}_i$ with $S_\alpha\subseteq S$, $\mathscr{G}_i\subseteq\mathscr{G}$.

Let us first concentrate on the S symmetry. Clearly, S is itself an isotropy subgroup in Ξ, with fixed space $W(S)$ isomorphic to M: in fact, if $\xi=(x_1,\ldots,x_n)$ with $x_1=x_2=\cdots=x_n=\chi\in M$, then $\sigma\xi=\xi$, $\forall\sigma\in S$, and

$$\begin{aligned} W(S)&=\{\xi\in\Xi\,|\,\sigma\xi=\xi\ \forall\sigma\in S\}\\ &=\{\xi=(x_1,\ldots,x_n)\,|\,x_1=x_2=\cdots=x_n\}\simeq\{x\}=M \end{aligned}\tag{12}$$

Notice that if S is the only symmetry in the model (i.e., $G=\{e\}$), then $W(S)$ corresponds to the only critical direction.

Consider now the G action on M; let $\chi_0\in M$ correspond to a critical direction for it, with $G_{\chi_0}=H\subseteq G$. It is quite immediate that also $\xi_0=(\chi_0,\ldots,\chi_0)\in\Xi$ corresponds to a critical direction for the action of $S\otimes_{\rightarrow}\mathscr{G}$ on Ξ.

Indeed, $\mathscr{G}=(G)^{\otimes n}$ acts on $W(S)$ by the diagonal action, i.e., by $g\otimes\cdots\otimes g$, so that on the diagonal subspace $W(S)\simeq M$ the $\mathscr{G}$-action is just a G-action.

Now "generally" [but not always: see Field and Richardson (1989, 1990, 1992; Field, 1989) for recent complete discussion and results] critical directions correspond to maximal isotropy subgroups (MIS).[5,6] We have just

[4]We write it in this way because it would not be possible to write directly μ in terms of $(\lambda-\lambda_*)$; the perturbative techniques needed to write s and μ as a series in ε go back to Poincaré and Lindestedt; see, e.g., Sattinger (1973).

[5]It should be remarked that this is, e.g., the case for the adjoint representation of any $SU(n)$ group, and for a number of other physically interesting cases; the determining criterion is based on Weyl groups (Field and Richardson, 1989, 1990, 1992; Field, 1989).

[6]Since we want to deal with linear representations, we always have that $x=0$ is invariant under the full group, which is therefore an isotropy subgroup. We define a nontrivial isotropy subgroup as a subgroup H such that $W(H)\backslash\{0\}$ is not empty, and by an MIS we will mean a maximal nontrivial isotropy subgroup.

seen that S is itself an isotropy subgroup, with nontrivial fixed space; therefore the MIS of G_V will be of the form

$$G^i_V = S \otimes_{\rightarrow} \mathscr{G}_i \tag{13}$$

with $\mathscr{G}_i \subseteq \mathscr{G}$. This has S as a subgroup, and therefore $W(G^i_V) \subseteq W(S)$, so that every MIS of G^i_V has a subspace of $W(S)$ as a fixed space.

We have already remarked that $\mathscr{G}$ acts on $W(S) \simeq M$ by the diagonal action, i.e., as G, so that finally we have that the MIS of G_V are necessarily of the form

$$G^i_V = S \otimes_{\rightarrow} (G_i^{\otimes n})_{\text{diag}} \simeq S \otimes_{\rightarrow} G_i \tag{14}$$

with $G_i \subseteq G$ an MIS for the action of G on M.

Let us summarize our discussion as follows:

(a) Critical directions under the G-action are reflected in critical directions under the $(S \otimes_{\rightarrow} \mathscr{G})$-action.
(b) Generically, critical directions of $(S \otimes_{\rightarrow} \mathscr{G})$ lie in the diagonal subspace $W(S) \simeq M$.
(c) On this subspace, critical directions correspond to those of the G-action; these are the $(S \otimes_{\rightarrow} \mathscr{G})$ critical directions whose existence is ensured by (a).

In physical terms, these points have the following meaning:

(a) All the symmetry-breaking patterns predicted by the standard (G) model will also appear in the Nielsen model.
(b) Generically, in the low-energy limit, Nielsen models $(S \otimes_{\rightarrow} \mathscr{G})$ break down their symmetry to the diagonal action, i.e., to the underlying (standard) G-model.
(c) When breaking down to the diagonal action, the Nielsen model predicts exactly the same symmetry-breaking patterns as the underlying (standard) G-model.

We stress once more that, depending on the underlying G symmetry, the Nielsen model can or cannot give symmetry breakings which are not predicted by the (standard) G-model; in any case the full symmetry-breaking content of the G-model is conserved by the corresponding Nielsen model.

We will now briefly consider some examples of Nielsen models with elementary G groups in order to illustrate our discussion.

6. SIMPLE EXAMPLES

We will adapt the Nielsen setting to the examples considered in Section 3, i.e., we consider now the corresponding Nielsen models.

(i) Here $M=R$, $G=Z_2$. This means that $V=V(x_1, \ldots, x_n)$ can actually be written as $V=V(x_1^2, \ldots, x_n^2)$; the S symmetry ensures that $V(\sigma\xi)=V(\xi)$, $\forall\sigma\in S$, where $\xi=(x_1, \ldots, x_n)$.

The Ξ space is R^n; the orbit space under $\mathcal{G}=(G)^{\otimes n}$ is R_+^2; taking into account the S symmetry, this reduces to the sector identified by, say, $x_1 \le x_2 \le \cdots \le x_n$.

The points on the line $x_1=x_2=\cdots=x_n=\chi$ have isotropy S for $\chi\neq 0$, and $S \otimes_{\rightarrow} Z_2$ for $\chi=0$; these are the only points with S symmetry, and therefore this line corresponds to a critical direction: the Nielsen model breaks down to a Z_2-model.

(ii) Let us consider the next example in full detail. Let $n=3$ and $G=SO(2)$ acting in R^3 as in Section 3; let $M=S^2\subset R^3$. We have seen that G-orbits on M are indexed by a real number $\omega\in[-1, 1]$, and there are two strata: $\Omega_0=\{-1\}\cup\{1\}$ with $G_x=SO(2)$ and $\Omega_1=(-1, 1)$ with $G_x=\{e\}$.

The orbit space of $\mathcal{G}=G\otimes G\otimes G$ can be represented as a cube; the vertices have $\mathcal{G}_x=SO(2)\otimes SO(2)\otimes SO(2)$, the edges have $\mathcal{G}_x=SO(2)\otimes SO(2)\otimes\{e\}$ (or permutation of the factors), the faces have $\mathcal{G}_x=SO(2)\otimes\{e\}\otimes\{e\}$ (or permutation of the factors), and the interior of the cube has $\mathcal{G}_x=\{e\}\otimes\{e\}\otimes\{e\}$.

If now we introduce the S_3 symmetry, the orbit space reduces to a sector of the cube, identified by, say, $x_1\le x_2\le x_3$; see Figure 1.

Now, the diagonal AD in the interior of the cube corresponds to points with $x_1=x_2=x_3$, i.e., $S_x=S_3$; the planes ABD and ACD in the interior of

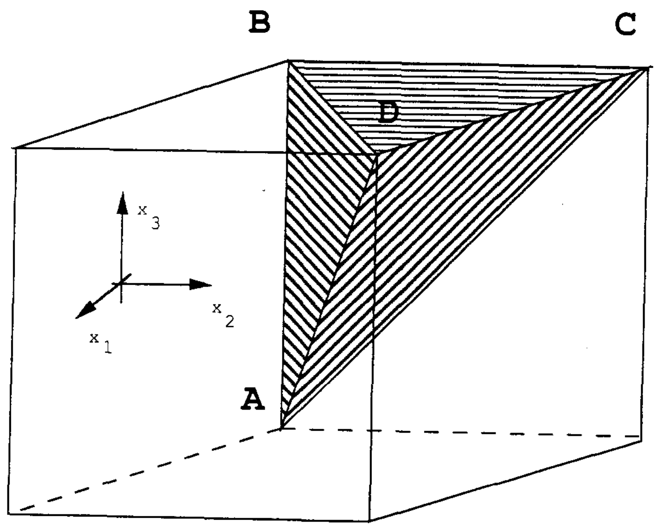

Fig. 1

Table I

Points	Relations satisfied	G_x
$x=A, D$	$x_1=x_2=x_3=\pm1$	$S_3 \otimes_\rightarrow [SO(2)]_d^3$
$x\in AD$	$x_1=x_2=x_3\neq\pm1$	$S_3 \otimes_\rightarrow \{e\}$
$x=B$	$x_1=x_2=-1, x_3=1$	$S_2 \otimes_\rightarrow [(SO(2))_d^2 \otimes SO(2)]$
$x=C$	$x_1=-1, x_2=x_3=1$	$S_2 \otimes_\rightarrow [SO(2) \otimes (SO(2))_d^2]$
$x\in AB$	$x_1=x_2=-1, x_3\neq\pm1$	$S_2 \otimes_\rightarrow [(SO(2))_d^2 \otimes \{e\}]$
$x\in AC$	$x_1=-1, x_2=x_3\neq\pm1$	$S_2 \otimes_\rightarrow [(SO(2)) \otimes (\{e\})^2]$
$x\in BC$	$x_1=-1, x_2\neq\pm1, x_3=1$	$\{e\} \otimes_\rightarrow [SO(2) \otimes \{e\} \otimes SO(2)]$
$x\in DC$	$x_1\neq\pm1, x_2=x_3=1$	$S_2 \otimes_\rightarrow [\{e\} \otimes (SO(2))_d^2]$
$x\in BD$	$x_1=x_2\neq\pm1, x_3=1$	$S_2 \otimes_\rightarrow [(\{e\})^2 \otimes SO(2)]$
$x\in ABD$	$x_1=x_2\neq\pm1, x_3\neq\pm1$	$S_2 \otimes_\rightarrow [(\{e\})^2 \otimes \{e\}]$
$x\in ACD$	$x_1\neq\pm1, x_2=x_3\neq\pm1$	$S_2 \otimes_\rightarrow [\{e\} \otimes (\{e\})^2]$
$x\in BCD$	$x_1\neq x_2\neq x_3, x_1, x_2\neq\pm1, x_3=1$	$\{e\} \otimes_\rightarrow [(\{e\})^2 \otimes SO(2)]$
$x\in$interior	$x_1\neq x_2\neq x_3, x_1, x_2, x_3\neq\pm1$	$\{e\} \otimes_\rightarrow [\{e\}]$

the cube correspond to points with, respectively, $x_1=x_2$ and $x_2=x_3$ and therefore $S_x=S_2$. Considering the full $S \otimes_\rightarrow \mathscr{G}$ symmetry, we have the situation depicted in Table I. In terms of strata, we have the situation described in Table II.

Notice that not only the vertices A and D corresponding to the full symmetry $S_3 \otimes_\rightarrow SO(2)$, but also the vertices B and C are isolated in their stratum and therefore correspond to critical directions. In other words, besides the breaking to the underlying $SO(2)$ model, the present example shows other symmetry breakings. Notice that these correspond to different copies of M selecting different (equivalent) critical points $\omega=\pm1$.

(iii) Let now $M=R^2$, $G=Z_2^x \otimes Z_2^y$, so that the G-orbit space is R_+^2. Again, for any n the $W(S)$ plane $x_1=x_2=\cdots=x_n=\chi\in R^2$ will have S_n symmetry; in this plane we can repeat the analysis of Section 3.

Notice that since there is only one G-orbit isolated in its stratum, the only $(S_n \otimes_\rightarrow \mathscr{G})$-orbit isolated in its stratum will lie in $W(S)$: it is $x_1=\cdots=x_n=0$.

Table II

Stratum Σ	Elements of Σ	G_x
$\Omega_0^{(1)}$	$\{A, D\}$	$S_3 \otimes_\rightarrow SO(2)$
$\Omega_0^{(2)}$	$\{B, C\}$	$S_2^{[1,2]} \otimes_\rightarrow [(SO(2))_d \otimes SO(2)]$
$\Omega_1^{(1)}$	$\{AB, CD\}$	$S_2 \otimes_\rightarrow SO(2)$
$\Omega_1^{(2)}$	AD	$S_3 \otimes_\rightarrow \{e\}$
$\Omega_1^{(3)}$	$\{AC, BD\}$	$S_2 \otimes_\rightarrow SO(2)$
$\Omega_1^{(4)}$	BC	$\{e\} \otimes_\rightarrow [SO(2) \otimes SO(2)]$
$\Omega_2^{(1)}$	$\{ABD, ACD\}$	$S_2 \otimes_\rightarrow \{e\}$
$\Omega_2^{(2)}$	BCD	$\{e\} \otimes_\rightarrow SO(2)$
$\Omega_3^{(1)}$	$\{\text{interior}\}$	$\{e\}$

Should we take $M=S^1\subset R^2$, we would have three strata for G-orbits: one given by points (0, 1) and (0, −1) with isotropy Z_2^x, one by points (1, 0) and (−1, 0) with isotropy Z_2^y, and the other one by remaining points with isotropy $\{e\}$.

Now, considering, e.g., $n=2$, not only are the (orbits of the) points with $x_1=x_2=(0,1)$ or $(1,0)$ isolated in their stratum, but so is (the orbit of) $x_1=(1,0)$, $x_2=(0,1)$.

(iv) Finally, let us consider $G=SU(2)$ with the standard action on $C^2\simeq R^4$. This action is such that $G_x=SO(2)=U(1)$ for $x\neq 0$, $G_x=SU(2)$ for $x=0$. Therefore the only G_V-orbits which are isolated in their stratum are those lying in $W(S)$, i.e., corresponding to the diagonal $SU(2)$ action.

The same holds for the standard action of $SU(3)$ on C^3 or, by the way, of $SU(n)$ in C^n or $SO(n)$ in R^n ($n\geq 2$), the relevant property being the transitivity of the action on the relevant unit sphere, so that all points $x\neq 0$ belong to the generic stratum.

This does not hold for the adjoint $SU(n)$ action, $n\geq 3$: e.g., for $SU(3)$ it is well known (Michel and Radicati, 1971, 1973) that there are nontrivial directions [corresponding to the $SU(3)$ octet]; in analogy to case (ii), one will have, besides the breaking to diagonal action, other possible symmetry breakings. To ensure the diagonal one is favored over the other ones, one could still find suitable conditions on the potential: e.g., in (7) a term with $\beta>0$.

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